Cosmological dynamics in six-order gravity

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Abstract

We consider cosmological dynamics in generalized modified gravity theory with the $R \Box R$ term added to the action of the form $R + R^N$. Influence of $R \Box R$ term to the known solutions of modified gravity is described. We show that in particular case of $N = 3$ these two non-Einstein terms are equally important on power-law solutions. These solutions and their stability have been studied using dynamical system approach. Some results for the case of $N \neq 3$ (including stability of de Sitter solution in the theory under investigation) have been found using other methods.

1 Introduction

Theories of modified gravity become very popular in the beginning of our century and the main reason is attempts to understand accelerated expansion of the Universe which is proved observationally [1, 2]. Modifying left-hand side of Einstein equations of gravity it is in principal possible to obtain a late-time acceleration without invoking any new type of matter. It is also known that such theories can explain inflation (i.e. accelerated expansion at early stage of Universe evolution) using only gravitational sector of the theory [3, 4], though inflation driven by the scalar field without any corrections to Einstein gravity is much popular now (nevertheless, inflation from modified gravity is not currently ruled out by observations [5]).

Standards way to modify Einstein gravity is to replace the curvature scalar R in the gravity Lagrangian by some scalar function $f(R)$. In principal, it is possible to consider Lagrangians which depends on other curvature invariants, like the Kretchmann invariant $R_{ijkl}R^{ijkl}$ or Ricci square $R_{ik}R^{ik}$ etc. However, cosmological evolution in such theories often lead to anisotropic expansion [6, 7, 8] and, therefore, can not describe late stages of our Universe. This is the main reason why nowadays the class of $f(R)$ theories is the most popular class of modified gravity, and we restrict ourselves by function of scalar curvature invariant in the present paper (it should be noted, however, that theories with scalar function $f(G)$ of Gauss-Bonnet invariant $G = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$ are promising due to special properties of the invariant G [9, 10]).

Apart from considering scalar function of R it is possible to introduce terms depending on derivatives of the scalar curvature. These theories are known from the end 80th, in

particular, theories with terms which are polynomial in the combination $R\square R$ have been intensively investigated [11, 12, 13, 14, 15, 16].

The goal of the present paper is to consider a theory which contains *both* $f(R)$ and $R\square R$ modifications of the Einstein gravity. We concentrated our attention on the liner with respect to $R\square R$ term case, because as it was shown only liner case is the ghost free one [17]. We apply theory of dynamical system to study asymptotic regimes and their stability. This method is used actively for finding asymptotic solutions in $f(R)$ gravity and studying their stability [18, 19, 20], we extend it for theories with $R\square R$ term. We show that this approach works well in the situation when corrections from these two ways of modifying gravity are equally important. One such case is identified ($f(R) = R + \beta R^3$) and investigated systematically. After, we discuss some problems of the used method in a general situation. Finally, some modifications of known cosmological behavior in $f(R)$ theories due to presence of $R\square R$ term are described for a broader class of functions $f(R)$.

2 Main equations

We consider Lagrangian in the form

$$L = \sqrt{-\mathrm{g}}f(R, R\square R \equiv A) + L_M. \tag{1}$$

Equations of motion for such kind of theories have the form

$$-\tfrac{1}{2}f\mathrm{g}_{ik} + f_R R_{ik} - \nabla_i\nabla_k f_R + \mathrm{g}_{ik}\square f_R+$$

$$+f_A\square R R_{ik} + \square(f_A R)R_{ik} - \nabla_i\nabla_k(f_A\square R + \square(f_A R)) + \square(f_A\square R + \square(f_A R))\mathrm{g}_{ik}+ \tag{2}$$

$$+\tfrac{1}{2}\nabla_l(f_A R)\nabla^l R\mathrm{g}_{ik} - \nabla_i(f_A R)\nabla_j R + \tfrac{1}{2}f_A R\square R\mathrm{g}_{ik} = 8\pi G T_{ik},$$

and its trace

$$-2f + f_R R + 3\square f_R + Rf_A\square R + R\square(f_A R)+$$
$$\tag{3}$$
$$+3\square(f_A\square R + \square(f_A R)) + \nabla_l(f_A R)\nabla^l R + 2f_A R\square R = 8\pi G T,$$

where $f_R = \frac{\partial f}{\partial R}$, $f_A = \frac{\partial f}{\partial A}$, $\nabla_i R = \frac{\partial R}{\partial x^i}$. Since we interested in the flat Friedmann-Robertson-Walker metrics

$$ds^2 = dt^2 - a^2(t)dl^2, \tag{4}$$

the following simple relations can be written down

$$T = \rho - 3p, \ T_{00} = \rho, \ R = -6(2H^2 + \dot{H}), \ R_{00} = \frac{1}{2}R + 3H^2, \ \square R = 3H\dot{R} + \ddot{R}. \tag{5}$$

Denoting terms depending on R and $R\square R$ as f_1 and f_2 respectively, we get

$$f(R, R\square R \equiv A) = f_1(R) + f_2(R\square R) \tag{6}$$

As it was already noted in the present paper we study the liner case of $f_2(R\Box R) = \alpha R\Box R$, the function $f_1(R)$ will be specified later. It is useful to introduce the following notations: $\partial f_1/\partial R = F$ and $\alpha \Box R = \Phi$. As F and Φ are scalars, in analog with eq. (5) we get:

$$\Box F = 3H\dot{F} + \ddot{F}, \quad \Box\Phi = 3H\dot{\Phi} + \ddot{\Phi}. \tag{7}$$

Taking into account this relations (00) component of eq. (2) may by written as

$$-\frac{1}{2}f_1 + \frac{1}{2}RF + 3H^2F + 3H\dot{F} + 2(\frac{1}{2}R + 3H^2)\Phi + 6H\dot{\Phi} - \frac{\alpha}{2}\dot{R}^2 = K\rho, \tag{8}$$

and its trace (3) take the form

$$-2f_1 + FR + 9H\dot{F} + 3\ddot{F} + 2\Phi R + 18H\dot{\Phi} + 6\ddot{\Phi} + \alpha\dot{R}^2 = K\rho(1 - 3w), \tag{9}$$

where we denote $K = 8\pi G$ and $p = w\rho$.

For further investigations we introduce the following dimensionless variables

$$x = -\frac{f_1}{6FH^2},$$

$$y = \frac{R}{6H^2},$$

$$z = \frac{\dot{F}}{FH}$$

$$m = \frac{2\Phi}{F}, \tag{10}$$

$$n = \frac{2\dot{\Phi}}{FH},$$

$$r = -\frac{\alpha\dot{R}^2}{6FH^2},$$

$$\Omega = \frac{K\rho}{3FH^2}.$$

Dividing eq. (8) by $3FH^2$, we rewrite it in our new variables (10):

$$x + y + z + 1 + m(y + 1) + n + r = \Omega. \tag{11}$$

Similarly, eq. (9) can be rewritten as

$$4x + 2y + 3z + \frac{\ddot{F}}{FH^2} + \frac{2\ddot{\Phi}}{FH^2} + 2my + 3n - 2r = \Omega(1 - 3w). \tag{12}$$

Taking derivative of the introduced variables with respect to time we get the system of equations

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$$\frac{\dot{x}}{H} = -\frac{\dot{R}}{6H^3} - xz - 2x\frac{\dot{H}}{H^2},$$

$$\frac{\dot{y}}{H} = \frac{\dot{R}}{6H^3} - 2y\frac{\dot{H}}{H^2},$$

$$\frac{\dot{z}}{H} = \frac{\ddot{F}}{FH^2} - z^2 - z\frac{\dot{H}}{H^2},$$

$$\frac{\dot{m}}{H} = n - mz, \tag{13}$$

$$\frac{\dot{n}}{H} = \frac{2\ddot{\Phi}}{FH^2} - nz - n\frac{\dot{H}}{H^2},$$

$$\frac{\dot{r}}{H} = -\frac{\alpha\dot{R}\ddot{R}}{3FH^3} - rz - 2r\frac{\dot{H}}{H^2},$$

$$\frac{\dot{\Omega}}{H} = \frac{K\dot{\rho}}{3FH^3} - \Omega z - 2\Omega\frac{\dot{H}}{H^2}.$$

To complete transformation we need to express all derivatives in (13) such as $\dot{H}$, $\dot{R}$ and other through new variables (10). First of all by using the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{14}$$

we find $\frac{K\dot{\rho}}{3FH^3} = -3\Omega(1 + w)$. Moreover since $R = -6(2H^2 + \dot{H})$, it easy to find that $\frac{\dot{H}}{H^2} = -2 - y$. Let us now introduce dimensionless parameters, characterizing the function $f_1(R)$:

$$a = \frac{Rf_{1R}}{f_1} = \frac{-y}{x},$$

$$b = \frac{Rf_{1RR}}{f_{1R}} = \frac{R\dot{F}}{F\dot{R}}, \tag{15}$$

$$c = \frac{Rf_{1RRR}}{f_{1RR}}.$$

Using these parameters we easily find

$$\frac{\dot{R}}{6H^3} = \frac{\dot{R}}{6H^3}\frac{F}{R\dot{F}}\frac{F\dot{F}}{F} = \frac{yz}{b}.$$

Using (5) and (7) it is possible to calculate

$$\frac{\ddot{F}}{FH^2} = \frac{z^2c}{b} - \frac{z^2ym}{2br} - 3z.$$

Using expression (12) we find

$$\frac{2\ddot{\Phi}}{FH^2} = -\frac{z^2c}{b} + \frac{z^2ym}{2br} - 4x - 2y - 2ym - 3n + 2r + \Omega(1 - 3w).$$

And finally, using $\ddot{R} = \frac{\Phi}{\alpha} - 3H\dot{R}$ we find

$$-\frac{\alpha\dot{R}\ddot{R}}{3FH^3} = -\frac{yzm}{b} - 6r.$$

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Note now that using eq. (11) it is possible to exclude Ω from resulting system. And summing all above expressions and going to the conformal time, we find resulting system:

$$x' = -\frac{yz}{b} + x(2y + 4 - z),$$

$$y' = \frac{yz}{b} + 2y(2 + y),$$

$$z' = \frac{z^2 c}{b} - \frac{z^2 ym}{2br} - z^2 + z(y - 1),$$

$$m' = n - mz, \tag{16}$$

$$n' = -\frac{z^2 c}{b} + \frac{z^2 ym}{2br} - 4x - 2y - 2ym+$$

$$+2r + (x + y + z + 1 + m(y + 1) + n + r)(1 - 3w) + n(y - z - 1),$$

$$r' = -\frac{yzm}{b} + r(2y - 2 - z),$$

where we denote $' = \frac{d}{d(\ln a)}$. For some functions f_1 there are additional relations between x and y, a well as between z and r. In the next section we consider an example for which such relations exist.

3 The case of $f_1 = R + \beta R^3$.

In this special case some simplification can be done. First of all we note that the parameter $c = 1$. Moreover, the parameter b is a combination of variables x and y:

$$b = 3\frac{x + y}{y}. \tag{17}$$

It is possible also to find relation between r and z in this special case:

$$r = -\frac{\alpha z^2 y}{108\beta(x + y)}. \tag{18}$$

So we can substitute this relation into (16) and exclude r from the system:

$$x' = -\frac{y^2 z}{3(x+y)} + x(2y + 4 - z),$$

$$y' = \frac{y^2 z}{3(x+y)} + 2y(2 + y),$$

$$z' = \frac{z^2 y}{3(x+y)} + 18\frac{\beta ym}{\alpha} - z^2 + z(y - 1),$$

$$m' = n - mz, \tag{19}$$

$$n' = -\frac{z^2 y}{3(x+y)} - 18\frac{\beta ym}{\alpha} - 4x - 2y - 2ym - \frac{\alpha z^2 y}{54\beta(x+y)}+$$

$$+(x + y + z + 1 + m(y + 1) + n - \frac{\alpha z^2 y}{108\beta(x+y)})(1 - 3w) + n(y - z - 1).$$

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As we have two coupling constants, α and β our strategy is to find stationary points and corresponding eigenvalues separately for $\alpha = 1$ and $\alpha = -1$ assuming $\beta = 1$. Having this information we then consider the general case.

We start with $\alpha = 1$. Solving the system (19) with vanishing left-hand sides, we find the following stationary points.

1. (x, y, z, m, n, Ω):

$$(1, -2, 0, 0, 0, 0) \tag{20}$$

Corresponding eigenvalues (which have been found numerically) are

$$(-3 - 3w, -7.9040, 4.9040, -3.3677, 0.3677) \tag{21}$$

This is de Sitter point, and we can see that it is a saddle.

2.

$$\left(\frac{1}{2} - \frac{1}{6}w, \frac{1}{2}w - \frac{3}{2}, -2 - 2w, \frac{(w^2 - 1)}{3(w - 3)}, \frac{2(w^2 - 1)(1 + w)}{3(3 - w)}, \frac{(5w^3 + 22w^2 - 34w - 63)}{9(3 - w)} \right) \tag{22}$$

At least one of the eigenvalues is not negative

$$2(1 + w) \geqslant 0. \tag{23}$$

Other eigenvalues are

$$
\begin{aligned}
&5/4w - 1/4\pm \\
&1/4(-7w^2 - 50w + 313 + 8(-14w^4 - 2w^3 + 313w^2 - 624w + 387)^{1/2})^{1/2},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&5/4w - 1/4\pm \\
&1/4(-7w^2 - 50w + 313 - 8(-14w^4 - 2w^3 + 313w^2 - 624w + 387)^{1/2})^{1/2}.
\end{aligned}
$$

This is a matter dominated point in a high-curvature regime. This point is also a saddle. Scale factor a has the following dependence on time:

$$a(t) = a_0 |t - t_0|^{\frac{2}{w+1}}. \tag{25}$$

3.

$$(0.1623, -0.4870, -6.0520, -1.0625, 6.4305, 0) \tag{26}$$

Eigenvalues are (the last eigenvalue depends on equation of state parameter w, so we present values found for several particular w)

$$
(6.4850, -1.9200, 4.5650, 6.0520, \begin{cases} 9.0780, w = -1 \\ 7.5780, w = -0.5 \\ 6.0780, w = 0 \\ 4.5780, w = 0.5 \\ 3.0780, w = 1 \end{cases}) \tag{27}
$$

We can see that this point is a saddle. The scale factor is

$$a(t) = a_0|t - t_0|^{\frac{1}{1.5130}}. \tag{28}$$

4.
$$(1.3732, -4.1195, 8.4778, -1.0700, -9.0710, 0) \tag{29}$$

Eigenvalues are

$$(-13.5973, -8.0759, -8.4778, -5.5213, \begin{cases} -12.7167, w = -1 \\ -14.2167, w = -0.5 \\ -15.7167, w = 0 \\ -17.2167, w = 0.5 \\ -18.7167, w = 1 \end{cases}) \tag{30}$$

We can see that for all studied values of w this point is a stable node. The scale factor shows a phantom behavior.

$$a(t) = a_0|t - t_0|^{\frac{1}{-2.1195}}. \tag{31}$$

5.
$$(0.6979, -2.0936, 0.3742, -0.0326, -0.0122, 0) \tag{32}$$

Eigenvalues are

$$(4.7703, -0.3742, -8.2381, -0.3742, \begin{cases} -0.5613, w = -1 \\ -2.0613, w = -0.5 \\ -3.4678, w = 0 \\ -5.0613, w = 0.5 \\ -6.5613, w = 1 \end{cases}) \tag{33}$$

This is a phantom saddle with

$$a(t) = a_0|t - t_0|^{\frac{1}{-0.0936}}. \tag{34}$$

We now turn to the case of $\alpha = -1$. Similarly as before, we get the following stationary points:

1.
$$(1, -2, 0, 0, 0, 0) \tag{35}$$

with eigenvalues

$$(-3 - 3w, -3.4198, 0.4198, -1.5000 - 5.5844i, -1.5000 + 5.5844i) \tag{36}$$

This is de Sitter point which is a saddle.

2.
$$\left(\frac{1}{2} - \frac{1}{6}w, \frac{1}{2}w - \frac{3}{2}, -2 - 2w, \frac{1 - w^2}{3(w - 3)}, \frac{2(w^2 - 1)(1 + w)}{3(w - 3)}, \frac{(5w^3 - 8w^2 + 20w + 45)}{9(w - 3)}\right) \tag{37}$$

At least one of eigenvalues is not negative one

$$2(1 + w) \geqslant 0, \tag{38}$$

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others are

$$5/4w - 1/4\pm$$
$$1/4(-7w^2 + 94w - 119 + 8(-14w^4 + 34w^3 + 25w^2 - 300w + 1035)^{1/2})^{1/2}, \tag{39}$$

$$5/4w - 1/4\pm$$
$$1/4(-7w^2 + 94w - 119 - 8(-14w^4 + 34w^3 + 25w^2 - 300w + 1035)^{1/2})^{1/2}.$$

This is a matter dominated point which is a saddle with

$$a(t) = a_0|t - t_0|^{\frac{2}{w+1}}. \tag{40}$$

3.
$$(0.7023, -2.1069, 0.4275, 0.0374, 0.0160, 0) \tag{41}$$

Eigenvalues are

$$(-1.7672 + 5.7921i, -0.4275, -3.5343, -1.7672 - 5.7921i, \begin{cases} -0.6412, w = -1 \\ -2.1412, w = -0.5 \\ -3.5343, w = 0 \\ -5.1412, w = 0.5 \\ -6.6412, w = 1 \end{cases}) \tag{42}$$

For all studied values of w this point is an attractive focus. The behavior of the scale factor is of phantom type:

$$a(t) = a_0|t - t_0|^{\frac{1}{-0.1069}}. \tag{43}$$

All these points are summarize in the following two tables.

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Table 1. Stationary points for $f_1 = R + \beta R^3$ with $\beta = 1$, $f_2 = \alpha R \Box R$ with $\alpha = 1$.

Number of point	Coordinates of stationary points	Stability type	Time dependence of scale factor $a(t)$
1.	$x = 1,$ $y = -2,$ $z = 0,$ $m = 0,$ $n = 0,$ $\Omega = 0.$	Saddle	$a(t) = a_0 e^{\frac{t-t_0}{\sqrt{12}}}$
2.	$x = 1.3732,$ $y = -4.1195,$ $z = 8.4778,$ $m = -1.0700,$ $n = -9.07100,$ $\Omega = 0.$	Attractive node	$a(t) = a_0 |t - t_0|^{-0.4718}$
3.	$x = 0.1623,$ $y = -0.4870,$ $z = -6.0520,$ $m = -1.0625,$ $n = 6.4305,$ $\Omega = 0.$	Saddle	$a(t) = a_0 |t - t_0|^{0.6609}$
4.	$x = 0.6979,$ $y = -2.0936,$ $z = 0.3742,$ $m = -0.0326,$ $n = -0.0122,$ $\Omega = 0.$	Saddle	$a(t) = a_0 |t - t_0|^{-10.6724}$
5.	$x = \frac{1}{2} - \frac{1}{6}w,$ $y = -\frac{3}{2} + \frac{1}{2}w,$ $z = -2 - 2w,$ $m = \frac{1}{3}\frac{-1+w^2}{-3+w},$ $n = -\frac{2}{3}\frac{(-1+w^2)(1+w)}{-3+w},$ $\Omega = -\frac{1}{9}\frac{5w^3+22w^2-34w-63}{w-3}.$	Saddle	$a(t) = a_0 |t - t_0|^{\frac{2}{w+1}}$

Table 2. Stationary points for $f_1 = R + \beta R^3$, $\beta =$, $f_2 = \alpha R \Box R$ with $\alpha = -1$.

Number of point	Coordinates of stationary points	Stability type	Time dependence of scale factor $a(t)$
1.	$x = 1,$ $y = -2,$ $z = 0,$ $m = 0,$ $n = 0,$ $\Omega = 0.$	Saddle	$a(t) = a_0 e^{\frac{t-t_0}{\sqrt{12}}}$
2.	$x = 0.7023,$ $y = -2.1069,$ $z = 0.4275,$ $m = 0.0374,$ $n = 0.0160,$ $\Omega = 0.$	Attractive node	$a(t) = a_0 \lvert t - t_0 \rvert^{-9.3545}$
3.	$x = \frac{1}{2} - \frac{1}{6}w,$ $y = -\frac{3}{2} + \frac{1}{2}w,$ $z = -2 - 2w,$ $m = -\frac{1}{3}\frac{-1+w^2}{-3+w},$ $n = \frac{2}{3}\frac{(-1+w^2)(1+w)}{-3+w},$ $\Omega = -\frac{1}{9}\frac{5w^3-8w^2+20w+45}{w-3}.$	Saddle	$a(t) = a_0 \lvert t - t_0 \rvert^{\frac{2}{w+1}}$

It is important to compare these results with those found in the absence of f_2- term. For the function $f_1 = R + R^3$ these are three stationary points – de Sitter (which is an unstable node), a stable vacuum power-law solution, a solution with $a \sim t^{1/2}$ behavior and an unstable matter-dominated power-law solution (the latter case represents a situation where the power index in not changed due to the dalambertian term).

In the presence of the dalambertian term we have analogs of three of these four solution. It is interesting, that substituting in the general form of equations of motion (16) the function $f_1 = R^3$ and using $f_{1R} = 3R^2, f_{1RR} = 6R, f_{1RRR} = 6, b = 2, c = 1$, and, correspondingly, $y = -3x, r = -z^2\alpha/72$, which gives the equation of motion in the form

$$x' = \frac{3xz}{2} + x(-6x + 4 - z)$$

$$z' = -\frac{z^2}{2} - 54\frac{xm}{\alpha} - z(1 + 3x)$$

$$m' = n - mz \tag{44}$$

$$n' = -\frac{z^2}{2} + 54\frac{xm}{\alpha} + 2x + 6xm - \frac{\alpha z^2}{36} +$$

$$+(-2x + z + 1 + m(-3x + 1) + n - \frac{z^2\alpha}{72})(1 - 3w) - n(1 + 3x + z).$$

we get a bit different set of stationary points. As this theory can be treated as high-curvature limit of the $R + R^3$ theory when Einstein contribution can be neglected, we do not find de Sitter point. However, this is not the only difference. There are two additional solutions which have the following form for $\alpha = 1$ for vector (x, z, m, n, Ω):

$$(0, 0, -1, 0, 0) \tag{45}$$

with eigenvalues

$$(4, 1 - 3w, -1, -1) \tag{46}$$

and

$$\left(0, -2, -\frac{19}{18}, \frac{19}{9}, 0\right) \tag{47}$$

with eigenvalues

$$(3, 3 - 3w, -1, -1) \tag{48}$$

which correspond to $a \sim t^{1/2}$ behavior. Both solutions have $x = 0, y = 0$, and, so, they can not be found between solutions of Eqs. (19) due to the factor $x + y$ in denominator. This is the same reason why we do not find a solution corresponding to a Friedmann Universe in low-curvature limit of the theory $R + R^3$ – this is already remarked in .

For other values of α coordinates of one of these point changes, though the behavior $a \sim t^{1/2}$ and the eigenvalues remain unchanged which indicates that these two points are saddles.

Returning to other points, we can see that the difference between $\alpha = 1$ and $\alpha = -1$ cases is in number of power-law vacuum solutions – there are three of them for $\alpha = 1$ (with one stable solution) and one (which is stable) for $\alpha = -1$. We remind a reader that for $\alpha = 0$ we have one stable node $a \sim (t - t_0)^{-10}$. Using power-law ansatz we can study a general case.

Substituting $a(t) = a_0|t - t_0|^{\frac{1}{A}}$ into eq. (9),we get for $f_1 = R + \beta R^3$

$$\frac{1}{A^6(t-t_0)^6}(\beta(-6)^3(2 - A)^3 + 3(A^4(t - t_0)^4 + 3\beta(-6)^2(2 - A)^2) +$$

$$+18\beta(-6)12(2 - A)^2 A + \alpha(36(-6)(1 - A)^2(2 - A)A + \tag{49}$$

$$+6\alpha(-144)(2 - A)(1 - A)A^2 - 72\alpha(2 - A)^2 A^2 - K\rho_0 A^6(t - t_0)^{-\frac{3(1+w)}{A}+6}) = 0.$$

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Neglecting matter and Einstein term we get the equation for A

$$20\alpha A^3 + 2A^2(15\beta - 8\alpha) - A(57\beta + 6\alpha) - 6\beta = 0. \tag{50}$$

The solution is plotted in Fig.1. Stability analysis shows that one solution is stable, stable branches are shown in bold.

Other solutions do not show any significant dependence on the value of α. The coordinates of de Sitter point do not depend on α, and stability analysis shows that it is always a saddle. For the matter dominated point one eigenvalue is equal to $2 - 2w$ for any α, so this point is unstable (though dimension of unstable manyfold for this point does depend on α). Finally, eigenvalues of the points with $a \sim t^{1/2}$ does not depend on α.
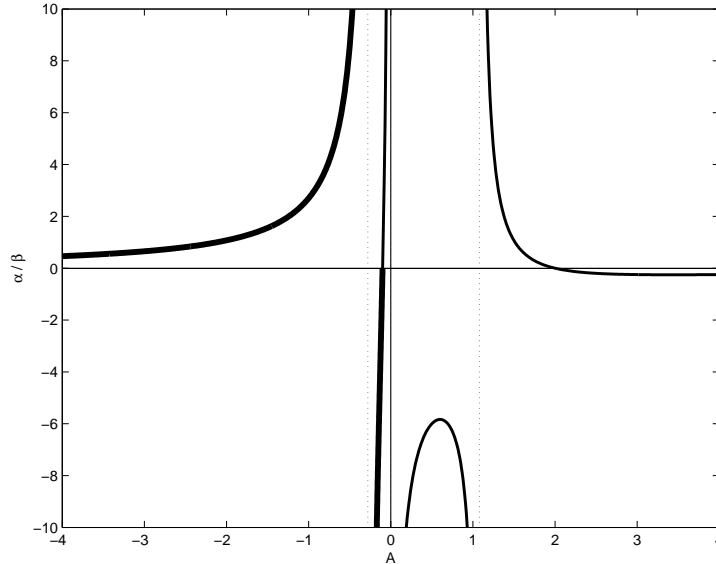

Figure 1: Power index A of a vacuum power-law solution depending on coupling constants α and β. Stable branches are bold lines.

4 The case of $f_1 = R + R^N, N \neq 3$

The equation (49) shows also why the case of $f_1 = R + R^3$ is exceptional for theories with $f_2 = R\Box R$. In this case for power-law ansatz $a \sim (t - t_0)^{1/A}$ terms originating from f_1 and f_2 are equally important. For functions $f_1 = R + R^N$ with $N > 3$ in the high-curvature regime terms from f_1 dominates (in low-curvature regime dominant terms are those from Einstein part of f_1). This means that influence of f_2 could be important in some transient regime of intermediate energy, and, so, it can not be found by studies of stable points and corresponding asymptotic regimes.

On the other hand, for $1 < N < 3$ the evolution of the Universe begins with $R\Box R$ dominant regime, after that R^N corrections can be important in an intermediate regime, and for low-curvature stage Einstein term becomes dominant. To study such type of a transient dynamics we need other methods.

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In what follows we consider two particular problems which can be solved using a specially chosen anzatz.

4.1 Stability of de Sitter solution

It is known that in a modified gravity with $f_1 = R + R^N$ de Sitter solution exists and stable for $0 < N < 2$, $N \neq 1$. We study what happens if we include the dalambertian term.

In this section it will be better for us to use the different signature from previous one. Thus we put

$$\mathrm{g}_{ik} = diag(-1, a^2, a^2, a^2) \tag{51}$$

In this notation some values change the sign, for instance we have $R = 6(\dot{H} + 2H^2)$, $R_{00} = 3(\dot{H} + H^2)$ and respectively $\Box R = -\ddot{R} - 3H\dot{R}$.

We take function f in the form:

$$f = R + \beta R^N + \alpha R \Box R. \tag{52}$$

Thus 00 component of (2) takes the form:

$$6H^2 + \beta[(1-N)R^N + 6H^2 N R^{N-1} + 6HN(N-1)R^{N-2}\dot{R}]$$

$$+\alpha[2R\ddot{R} + 36H^3\dot{R} - \dot{R}^2 - 48H^2\ddot{R} - 12H\dddot{R}] = 0. \tag{53}$$

This representation is more comfortable for following investigations. There is a dS-solution in the equation (53):

$$R_0^{N-1} = \frac{1}{\beta(N-2)}. \tag{54}$$

Note that here we have $R_0 = 12H_0^2$. We can see that there is no dS-solution for $N = 1$ in empty space, as well as there is no dS-solution for $N = 2$ in empty space. For $N = 3$ we have dS-solution $H_0^2 = 1/(12\sqrt{\beta})$. Also its need to note that dS-solution exist for positive β when $N > 2$ and for negative one when $N < 2$. This result is the same as for usual $f(R)$-gravity, because additional term $\alpha R \Box R$ can not produce new dS-solution since it contain only curvature derivative, however it may change the stability of dS-solutions due to terms with higher time derivatives.

For finding stability conditions we study small perturbations near dS-solution H_0. We note that $H = H_0 + \delta H$, $R = R_0 + \delta R = R_0 + 6(\delta\dot{H} + 4H_0\delta H)$ and $R^N = R_0^N + 6NR_0^{N-1}(\delta\dot{H} + 4H_0\delta H)$, where we keeping only linear with respect to δH terms. Substituting this relations into (53) and keeping only linear terms with respect to δH terms and using (54) we find

$$24H_0\alpha\delta H^{(4)} + 12\alpha R_0\delta\dddot{H} + \left(10H_0 R_0\alpha - \frac{N(N-1)}{N-2}H_0^{-1}\right)\delta\ddot{H}$$

$$-\left(2R_0^2\alpha + 3\frac{N(N-1)}{N-2}\right)\delta\dot{H} + 4H_0(N-1)\delta H = 0. \tag{55}$$

We search for solutions of this equation taken in the form $\delta H = e^{\lambda t}$. After substituting this representation into equation (55) it transforms to usual algebraical equation of forth

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order with four solution λ_i (in the general complex case). It is clear that for stability solution H_0 need $Re(\lambda_i) < 0$ for any i. This algebraical equation is rather complicate one, so we use a special method (Routh-Hurwitz theorem), which detects if any root of this equation have a positive real part. For details see, for instance [22]. Briefly it works as follows: from coefficients of equation (55) we construct some determinants T_i and if all them are positive then all real part of solutions λ_i are negative. Here we present the results of calculations of T_i:

$$T_0 = 24H_0\alpha, \tag{56}$$

$$T_1 = 144H_0^2\alpha, \tag{57}$$

$$T_2 = 72H_0\left(12 \cdot 28H_0^4\alpha^2 - \alpha\frac{N(N-1)}{N-2}\right), \tag{58}$$

$$T_3 = 9 \cdot 24H_0\left(-12^2 \cdot 14 \cdot 16\alpha^3H_0^8 - 48\alpha^2H_0^4\frac{N-1}{N-2}(13N-16) + \alpha\frac{N^2(N-1)^2}{(N-2)^2}\right), \tag{59}$$

$$T_4 = 4H_0(N-1)T_3. \tag{60}$$

Note also that for positive α we need condition $T_i > 0$, but for negative one we need $T_i < 0$. First of all we can see that for $N < 1$ T_3 and T_4 always have opposite sing and therefor dS-solution is unstable for any $N < 1$ (note that we interested in only $H_0 > 0$ case). However, for sufficiently small values of H_0 all terms with high order H_0 can be neglected and all T_i have the same sign in the range $1 < N < 2$. It mean that in this range dS-solution may be stable for sufficiently big $|\beta|$. This analytical result is confirmed by numerical investigation (see Fig.2).
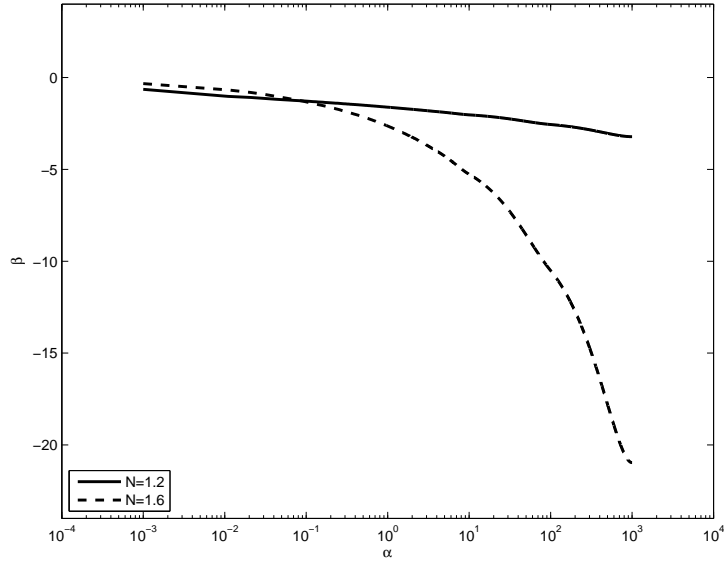

Figure 2: Zones of stability of de Sitter solution for two values of N. The de Sitter solution is stable below the curves.

4.2 Ruzmaikin solution and its analogs

The method we used have another limitation. Usually, having the variables determined we obtain the time dependence of the scale factor via Eqs. (10). The simplest way is to use equation for y. If the scale factor behavior is of power-law type, this equation gives us the exact solution. However, for other types of dynamics we can not be sure that the set of variables can distinguish asymptotic regimes, which are different from the cosmological point of view. An interesting example is the situation in modified gravity with $f_1 = R + R^2$. This particular case is an exceptional one in the sense that there are no de Sitter solution in this theory (see above). Instead, the Ruzmaikin solution $H \sim t$ exists [21]. From the cosmological perspective, this solution describing super-inflating Universe is very different from the de Sitter solution, however, it gives the same values of x, y, z as the latter one. Only by using initial equations (8)–(9) we can understand that something special happens for this particular function f_1. This explains why this solution have not been distinguished in papers [20, 18]. It is interesting that another set of variables used in [6] does discriminate between de Sitter and Ruzmaikin solutions.

In what follows we show that similar solution exists for the theory with $f_1 = R + R^2$ and $f_2 = R\Box R$. Consider

$$a(t) = const \cdot e^{\frac{Bt^{n+1}}{n+1}},$$
$$H = Bt^n. \tag{61}$$

we get

$$\dot{H} = Bnt^{n-1}, \tag{62}$$

$$R = -6(2B^2t^{2n} + Bnt^{n-1}), \tag{63}$$

$$\dot{R} = -6(4nB^2t^{2n-1} + Bn(n-1)t^{n-2}), \tag{64}$$

$$\ddot{R} = -6(4n(2n-1)B^2t^{2(n-1)} + Bn(n-1)(n-2)t^{n-3}), \tag{65}$$

$$\Box R = -72nB^3t^{3n-2} - 6n(11n-7)B^2t^{2(n-1)} - 6Bn(n-1)(n-2)t^{n-3}, \tag{66}$$

$$\frac{\partial \Box R}{\partial t} = -72n(3n{-}2)B^3t^{3(n-1)} {-} 12n(11n{-}7)(n{-}1)B^2t^{2n-3} {-} 6Bn(n{-}1)(n{-}2)(n{-}3)t^{n-4}. \tag{67}$$

Equations of motion in the absence of matter take the form

$$Et^{3n-1} + Mt^{2(n-1)} + St^{2n} +$$
$$+ D_1t^{5n-1} + E_1t^{4n-2} + S_1t^{3(n-1)} + M_1t^{2(n-2)} = 0, \tag{68}$$

where constant coefficients depend on coupling constants. We need two of them

$$S = 3B^2, \tag{69}$$

$$D_1 = 36 \cdot 12\alpha B^5 n.$$

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We denote the terms originating from the dalambertian by variables with subscripts. The biggest power indexes (which we should equate) belong to the terms St^{2n} and $D_1 t^{5n-1}$. This means that

$$
\begin{aligned}
2n &= 5n - 1, \\
3n &= 1, \\
n &= \tfrac{1}{3},
\end{aligned}
\tag{70}
$$

and we get the solution of Ruzmaikin type:

$$
\begin{aligned}
a(t) &= const \cdot e^{\frac{3Bt^{\frac{4}{3}}}{4}}, \\
H &= Bt^{\frac{1}{3}}.
\end{aligned}
\tag{71}
$$

As the original Ruzmaikin solution, this solution corresponds to (x, y, z, m, n, r):

$$
(1, -2, 0, 0, 0, 0)
\tag{72}
$$

and is undistinguishable from de Sitter point from the viewpoint of our set of dimensionless variables.

As the coefficient D_1 comes from the dalambertian contribution, this solution exists also in a theory with $f_1 = R$, $f_2 = R\Box R$, i.e. theory with only dalambertian correction to Einstein gravity.

As for power-law solutions, substituting $a(t) = a_0(t - t_0)^{\frac{1}{A}}$ we can obtain solutions valid in the regime when $R\Box R$ dominates. There are vacuum solutions with $A_1 = -0.2782$ and $A_2 = 1.0782$ (these solutions represent solutions of Eq.(50) in the limit $\alpha \to 0$), and matter-dominated solution $A = \frac{1}{2}(1 + w)$. These three solutions are valid in the high-energy regime for $f_1 = R + R^N$, $1 < N < 3$.

5   Conclusion

We have considered cosmological dynamics in the modified gravity with both $f(R)$ and $R\Box R$ terms in the action. In this theory the case of $f(R) = R + R^3$ appears to be a special one. In this particular case terms in the equations of motion originating from these two different modifications of Einstein gravity (namely, adding R^3 and $R\Box R$ to the action) are equally important on power-law solutions. This allows us to obtain several exact solutions and study their stability using dynamical systems approach. Our results show that the general picture does not change much in comparison with the known case of $R + R^3$ gravity – only one stationary point (which corresponds to a phantom solution) is stable, others (their number can vary) are unstable.

Other power-law functions $f(R) = R + R^N$ lead to zones of $f(R)$ dominance (near a cosmological singularity for $N > 3$), or $R\Box R$ dominance (R^N corrections are sub-dominant in comparison with those from the dalambertian near a cosmological singularity if $N < 3$), as well as existence of transient epoch when some terms can be temporarily important (R^N corrections for $N < 3$ and $R\Box R$ corrections for $N > 3$).

However, this does not mean that sub-dominant terms in some particular epoch are absolutely unimportant. The reason of this is extra derivatives in terms originating from

$R\Box R$ contribution. This means that some stable points existing in $R + R^N$ gravity can loose their stability due to $R\Box R$ contribution. We study in detail one particular example of this situation, namely stability of de Sitter solution. It is known that de Sitter solution exists and stable in $f(R)$ gravity with $f = R + R^N, N < 2\,(N \neq 1)$. $R\Box R$ terms vanishes on de Sitter, however, extra dimensions of the phase space may violate stability. We have shown that, indeed, for $N < 1$ the de Sitter solution is unstable (for $1 < N < 2$ the stability depends on coupling constants).

Finally, a modification of Ruzmaikin solution for $R + R^2$ gravity valid for theories with $R\Box R$ corrections is found.

Acknowledgments

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